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A Special Riemann Surface.*

BY H. H. CONWELL.

(Plates II to V.)

THE purpose of this paper is to consider in detail, for elliptic functions and briefly for hyper-elliptic functions, a special Riemann surface in three space obtained as the projection of the intersection of two hyper-surfaces in four space.

It will be seen that the surface investigated here is of advantage in the fact that it can be easily identified, from the point of view of analysis situs, with a double-faced disk having p holes; where $p = [\frac{n-1}{2}]$†, n being the degree of the function. In Riemann's real representation this is obtained only after an artificial and somewhat complicated dissection of the surface, in which the determination of the branch points is a very important factor. In a sense this difficulty may be said in our case to have been merely shifted from such a dissection to the construction of a certain real surface from its equation in three space. This construction can, however, be made very simple. In the ordinary Riemann surface the actual location of the branch points is difficult at best, and is useless so far as the investigations bearing on the surface are concerned. The actual construction of the surface under consideration will be avoided except in the simplest case, and then only as much of its outline as is necessary will be obtained. This construction will be found to be comparatively simple.

* Received for publication on April 29, 1920.

† $[\frac{n-1}{2}]$ is understood to mean the greatest integer in $\frac{n-1}{2}$.

Let $f(w, z) = O$ be an irreducible polynomial in the two complex variables w and z, with either real or imaginary constant coefficients. Substituting $w = u + iv$ and $z = x + iy$ in the above relation we obtain the equation,

$$P(x, y, u, v) + iQ(x, y, u, v) = 0 \quad \ldots\ldots\ldots \quad (1)$$

Whence,

$$P(x, y, u, v) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$Q(x, y, u, v) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

The last two equations represent real three dimensional manifolds in the real four space (x, y, u, v). Their intersection in four space will be the surface Φ. Assume that $w = w_0$ when $z = z_0$. It is then possible, in the neighborhood z_0, w_0, to expand $(w - w_0)$ in powers of $(z - z_0)$ and by analytical continuation to go from the neighborhood of z_0 to the neighborhood of z_1. As z changes from z_0 to z_1, w will change from w_0 into one of the values w_1 corresponding to z_1. If this process be continued until z by a continuous succession of values returns to z_0, w may or may not return to w_0. In the first case the representative point on Φ corresponding to a pair of values (w, z) will describe a closed path, while in the second case the path will be open. The obvious one to one correspondence between points of the surface Φ and sets of values (w, z) shows that this surface can play the same role as the ordinary Riemann surface.

If between equations (2) and (3) v is eliminated there arises the relation,

$$F(x, y, u) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

which represents in the three space (x, y, u), a surface F, viz., the projection of Φ in that space. This surface F, as well as Φ, can be used as a Riemann image, this being the configuration to be investigated in this paper. We shall limit ourselves, as before stated, to the hyper-elliptic case. It is evident that the x, y or u projection of Φ would serve the same purpose as F.

Before proceeding with the general cubic a special cubic will be considered in detail, and enough of the resulting surface constructed to show its properties as a Riemann image. (This special cubic is chosen on account of its adaptability to cross-section representation.)

Consider the equation

$$w^2 = z^3 - 31z - 30 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

from which

$$P \equiv u^2 - v^2 - (x^3 - 3xy^2 - 31x - 30) = 0 \quad \ldots \quad (6)$$

and

$$Q \equiv 2uv - (3x^2y - y^3 - 31y) = 0 \quad \ldots\ldots\ldots \quad (7)$$

The intersection of $P = 0$ and $Q = 0$ in four space is the surface Φ. The v projection of Φ in three space has for its equation

$$F(x, y, u) = 4u^4 - 4u^2 (x^3 - 3xy^2 - 31x - 30) - (3x^2y - y^3 - 31y)^2 = 0 \ldots\ldots \quad (8)$$

This surface is symmetric to both the XU and XY planes. The trace on the XU plane is the XX axis and the real curve

$$u^2 = x^3 - 31x - 30 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

representing all the real pairs (w, z) satisfying the original equation. The curve represented by (9) consists of an infinite branch and an oval (see fig. I). The XY trace consists of the XX axis and the hyperbola (see fig. II).

$$3x^2 - y^2 = 31 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

This hyperbola and the XX axis are the only double curves of the surface.

From equation (4) we obtain,

$$u = \pm \sqrt{\tfrac{2}{2}} \left[S + (S^2 + T^2)^{\frac{1}{2}} \right]^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

where

$$S = x^3 - 3xy^2 - 31x - 30 \quad \ldots\ldots\ldots\ldots\ldots \quad (12)$$

and

$$T = 3x^2y - y^3 - 31y \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

In this expression for u only positive values of the inner radical are considered as only real points on the surface F are to be investigated. Investigations of (11) show that when $y = 0$, $\frac{\delta u}{\delta y} = 0$ for all values of x except 6, -1 and -5, where it is infinite. For values of $x \leqq \sqrt{\frac{31}{3}}$ and $y > 0$, $\frac{\delta u}{\delta y}$ is positive or negative according to whether u is positive or negative, while for negative values of y it is positive or negative according to whether u is negative or positive. Hence for all sections of the surface parallel

to the $Y\,U$ plane, where $x \leqq \sqrt{\frac{31}{3}}$ there will be either both a maximum and minimum point, or a double point, for y equal zero and for no other finite value of y. For $x > \sqrt{\frac{31}{3}}$, there are other maximum and minimum points and double points, and the curves all pierce the $X\,Y$ plane along the curve represented by equation (10). From the preceding discussion and an inspection of equation (9) and figure I it is evident that the orthogonal projection of F upon the $X\,U$ plane will be nowhere within the oval, and hence that there is a hole in F for which the oval is the central section.

It is obvious that the surface F is composed of two sheets (see figs. I-VII) which hang together along the XX axis from $-\infty$ to -5, from -1 to $+6$ and pass through each other along the branch of the double curve $T = 0$ which lies to the right of the YY axis.

Sections parallel to the XU plane give curves composed of two branches which cut each other in points on one branch of the double curve $T = 0$ and nowhere else. Each branch continues to infinity and there unites parabolically with the other. The YU sections also unite parabolically at infinity, and hence the two sheets of the surface F merge into each other everywhere at infinity.

The surface F may be reduced to a double-faced disk with one hole as follows: For all values of $x > \sqrt{\frac{31}{3}}$ deform the surface by pulling the sheets through each other in such a way that instead of cutting in two distinct points on $T = 0$ for each value of x they will cut each other in two coincident points. This deformation will be continuous and approach zero in magnitude as x approaches $\sqrt{\frac{31}{3}}$ and will nowhere produce a tear in the surface. Having made this deformation, project the surface upon the XU plane and the result will be a double-faced disk with one hole.

Starting at a point P in sheet I and continuing in any direction on the surface F we can always return to P. This closed path may be all in sheet I or in both sheets I and II. It may or may not pass through or around the oval. In the latter case the circuit can always be reduced to zero while in the former it cannot be so reduced, unless there be an even number of

such passages and they be in opposite directions. Hence any closed circuit on F can be reduced to zero or to sums of multiples of two irreducible circuits. These facts show the elliptic function to be doubly periodic over F.

THE GENERAL ELLIPTIC CASE FOR WHICH $f(z)$ HAS REAL ROOTS.

We shall now extend the preceding discussion to a general elliptic function of the type

$$w^2 = z^3 - pz + q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

where p is positive and q either positive or negative, and where the roots of

$$z^3 - pz + q = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

are all real. It will be shown that the resulting surface $F(x, y, u) = 0$ has properties identical with those of the special case already investigated, if judged from the point of view of the investigations of this paper.

We obtain at once, as in the preceding case,

$$F(x, y, u) = 4u^4 - 4u^2S - T^2 = 0 \quad \ldots\ldots\ldots (16)$$

where

$$S = x^3 - 3xy^2 - px + q \quad \ldots\ldots\ldots\ldots\ldots (17)$$

and

$$T = 3x^2y - y^3 - py \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

The similarity of the XU and XY traces to those in the preceding case is obvious. From (16) we obtain,

$$\frac{\delta u}{\delta y} = \frac{\sqrt{2}\, y[-6x(S^2 - T^2)^{\frac{1}{2}} + 3x^4 + 6x^2y^2 - 6qx + 3y^4 + 4py^2 - p^2}{4\,(S^2 - T^2)^{\frac{1}{2}}\,[S + (S^2 + T^2)^{\frac{1}{2}}]^{\frac{1}{2}}}.$$

For $y = 0$, $\frac{\delta u}{\delta y} = 0$ for all values of x except the roots of $x^3 - px + q = 0$, where it is infinite. For all negative values of $x \leq \sqrt{\frac{p}{3}}$, $\frac{\delta u}{\delta y}$ is positive or negative for values of $y > 0$, according to whether u is positive or negative, and negative or positive for $y < 0$ according as u is positive or negative. Hence for all sections parallel to the YU plane, where $x \leq \sqrt{\frac{p}{3}}$ there will be a maximum and minimum point for y equal zero and for no other finite value of y. Since the sum of the roots of (15) are zero, at least one root must be negative and at least one positive. It is also evident that the

oval passes through the two smaller roots of (15). Let r_1, r_2, r_3, be the roots of (15), where $r_3 > r_2 > r_1$; then $r_1 + r_2 + r_3 = 0$ and $-r_1 r_2 r_3 = q$. From the last relation and the fact that $\frac{2}{3} p \sqrt{\frac{p}{3}} > q$ it is evident that $\frac{2}{3} p \sqrt{\frac{p}{3}} > 2 r_3^2$ and therefore $\sqrt{\frac{p}{3}} > r_2$; in other words, $x = \sqrt{\frac{p}{3}}$ does not lie within the oval.

For $x > \sqrt{\frac{p}{3}}$ there are other maximum and minimum points or double points than for y equal zero. As in the simpler case these sections are parabolic in nature.

These investigations show that this surface has no important characteristics, from our point of view, not common to the more special case and is therefore always reducible to a double-faced disk with one hole.

THE GENERAL ELLIPTIC CASE.

Up to this point the investigations have been confined to the type, $w^2 = z^3 - pz + q$, where p and q were both real, p positive and the roots all real. It will now be shown that no generality is lost by this restriction.

Consider the general elliptic case,

$$w^2 = f(z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

where

$$f(z) = a_0 (z - r_1)(z - r_2)(z - r_3)(z - r_4) \quad \ldots (21)$$

and a_0, r_1, r_2, r_3, r_4, are real or imaginary constants. The elliptic integral resulting from this form may by a well known transformation of $f(z)$ be made to depend upon an integral of the type,

$$g(z) = b_0 (z^3 - a_2 z - a_3).* \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

No generality is therefore lost by replacing $f(z)$ by $g(z)$. The constants of (22) may be positive or negative, real or imaginary. If a_2 and a_3 are arbitrarily changed the surface F will undergo a deformation. The only matter of interest in the present paper is whether such a deformation increases or decreases the number of holes in F. It is of course evident that if the number of holes is diminished as a_2 and a_3 assume the

* Boehm, Elliptische Functionen, Zweiter Teil, page 128.

values a^0_2 and a^0_3, that as a_2 and a_3 approach a^0_2 and a^0_3 in value, one or more holes in the surface must be continually decreasing in size in such a way that when a^0_2 and a^0_3 are reached the surface has a node at the point (x_0, y_0, u_0) on F and *vice versa*. If (x_0, y_0, u_0, v_0) is the corresponding point on Φ, the latter will also have a node at this point. Therefore corresponding to nodes on F are nodes on Φ. At such nodes the tangent hyper-planes to

$$P(x, y, u, v) = 0$$

and

$$Q(x, y, u, v) = 0$$

are coincident. In order to investigate the nature of F at such places write the equations of the tangent hyper-planes to P and Q at the point (x_0, y_0, u_0, v_0), and the conditions for their coincidence. The equations in question are,

$$(x-x_0)P'_{x_0} + (y-y_0)P'_{y_0} + (v-v_0)P'_{e_0} + (u-u_0)P'_{u_0} = 0, \quad (23)$$

and

$$(x-x_0)Q'_{x_0} + (y-y_0)Q'_{y_0} + (u-u_0)Q'_{u_0} + (v-v_0)Q'_{v_0} = 0 \ldots (24)$$

The conditions for these two hyper-planes to be coincident is that

$$\frac{P'_{x_0}}{Q'_{x_0}} = \frac{P'_{y_0}}{Q'_{y_0}} = \frac{P'_{u_0}}{Q'_{u_0}} = \frac{P'_{v_0}}{Q'_{v_0}}.$$

It is evident, however, from the relation

$$P(x, y, u, v) + iQ(x, y, u, v) = 0$$

that

$$P'_{x_0} = Q'_{y_0},\ P'_{y_0} = -Q'_{x_0},\ P'_{u_0} = Q'_{v_0}, \text{ and } P'_{v_0} = -Q'_{u_0}.$$

Hence

$$P^{2\prime}_{x_0} + Q^{2\prime}_{x_0} = 0, P^{2\prime}_{y_0} + Q^{2\prime}_{y_0} 0, = P^{2\prime}_{u_0} + Q^{2\prime}_{u_0} = 0 \text{ and } P^{2\prime}_{v_0} + Q^{2\prime}_{e_0} = 0$$

and therefore

$$Px'_0 = P'_{y_0} = P'_{u_0} = Pv_0 = Q'_{x_0} = Q'_{y_0} = Q'_{u_0} = Q'_{v_0} = 0.$$

In the above relations

$$P = u^2 - v^2 - s\ (x, y)$$

and

$$Q = 2uv - t\ (x, y),$$

therefore it follows that $u = 0$ and $v = 0$ and therefore that $g(z) = 0$. Moreover, since

$$P'_{x_0} + iQ'_{x_0} = 0 \text{ and } P'_{y_0} + iQ'_{y_0} = 0$$

it follows that

$$s'x_0 + it'x_0 = 0 \text{ and } s'y_0 + it'y_0 = 0.$$

Therefore $g'(z_0) = 0$, showing that z is a double root of $g(z) = 0$. It is evident therefore that the surfaces P and Q, and hence F, may be deformed in any way we please without affecting its analysis situs properties provided that during this deformation $g(z) = 0$ never acquires any double roots. These conditions allow a deformation that will change complex roots into real and unequal roots without any two roots becoming equal in the process. Hence we may in this manner transform $g(z)$ into $j(z)$, where the roots of $j(z)$ are real and unequal.

The above conclusions show that no generality is lost in considering the simpler case and thereby avoiding the difficult task of dealing with imaginary coefficients. The difficulty introduced by imaginary coefficients is that due to the lack of symmetry with respect to the XU plane.

It is evident now that the surface constructed from the simplest possible relation is sufficient for a complete exposition of the Riemann surface properties of the most general elliptic function.

A NUMERICAL EXAMPLE OF THE HYPER-ELLIPTIC CASE.

As an introduction to the general hyper-elliptic function we will consider briefly a simple numerical example of the same. The details of the surface F will be considered sufficiently to show that the preceding discussion can be applied in all its essential details to the higher form. For this purpose consider the equation

$$w^2 = (z-5)(z-1)(z+1)(z+2)(z+3).$$

The surface $F(x, y, u) = 0$ will be represented by

$$4u^4 - 4u^2S - T^2 = 0,$$

where

$$S = x^5 - 10x^3y^2 + 5xy^4 - 20x^3 + 60xy^2 - 30x^2 + 30y^2 + 19x + 30$$

and

$$T = 5x^4y - 10x^2y^3 + y^5 - 60x^2y + 20y^3 - 60xy + 19y.$$

The surface F is symmetric to the XU and XY planes. The trace on the XU plane is the XX axis and the real curve

$$v^2 = (x-5)(x-1)(x+1)(x+2)(x+3)$$

representing all the real pairs (w, z) satisfying the original equation. The latter consists of two ovals and an infinite branch. The trace on the XY plane is the double curve represented by the equation $T^2 = 0$. This curve is composed of the XX axis and four infinite branches which are hyperbolic in form and coaxial (see fig. VIII).

Sections parallel to the XU plane give rise to curves which have double points on the branches I and III of the double curve, as shown in the figure, and nowhere else. This is shown by an investigation of the value of S in the neighborhood of these branches. For the two branches to hang together or intersect each other, it is necessary that T be equal to zero and S be negative or zero. Every pair of values (x, y) on one of these infinite branches reduces T to zero, but none of these pairs on branch II or IV will cause S to be negative or zero. Therefore the two sheets of the surface F do not cut through each other along either of these branches. The two sheets hang together along the XX axis from $-\infty$ to -3, -2 to -1, from $+1$ to $+5$ and cut each other along the two branches I and III of $T = 0$. To prove, as in the elliptic case, that the two sheets never hang together for any finite value of y except zero would be very complicated, and so another method is employed. It is easily seen that any section parallel to the YU plane will give rise to a curve which has a number, say d, double points. But this curve is composed of two branches which intersect in d points in the XY plane. If d is odd the two branches are odd and hence each branch stretches off to infinity in both directions. If d is even, each branch is even and hence cuts the line at infinity in an even number of places and is accordingly a closed curve. In the first case (d odd) the XX axis must be composed of intersection points, while in the latter it is not. This leads to the conclusion that all sections which cut the curve $u = f(x)$, $y = 0$ give rise to even branches and all others to odd. Hence the former are always reducible to traces of the form, fig. V or fig. VI, while the latter are always reducible to branches of the form fig. VII. From this will follow, as in the elliptic case, that F is two-sheeted and contains two holes. By a deformation similar to the one described in the example of the elliptic case, it may be brought into the form of a double-faced disk with two holes. Hence all

closed circuits on F may be reduced to zero or to sums of multiples of four irreducible circuits.

Having considered the elliptic case in detail and investigated briefly a special hyper-elliptic function, we now proceed to the most general hyper-elliptic function, $w = R(z)$, where $R(z)$ is of degree n.

Forming the equation of the surface F in the usual manner, there arises the equation $F(x, y, u) = 0$, where F is of degree $2n$ in (x, y, u). $F(x, y, u) = 0$ may always be put in the form,

$$4u^4 - 4u^2S - T^2 = 0,$$

where S and T are polynomials in x and y of degree n. As has been shown in the preceding considerations, $R(z)$ may be assumed to have only real roots. Hence the surface F is symmetric to the XU and XY planes. The XU trace will consist of the XX axis and a curve representing all real pairs (w, z) satisfying the original equation. The latter curve will consist of one or two infinite branches, according to whether n is odd or even, and p ovals. The XY trace will be a double curve represented by $T = 0$ and consisting of the XX axis and a curve represented by an equation of degree $(n - 1)$. This double curve represents all the real double points of the surface F.

The surface F is composed of two sheets which hang together everywhere along the XX axis except for values of x which satisfy the equation $u = R(x), y = 0$, and cut each other along certain branches of the double curve $T = 0$. Corresponding to the p ovals there will be p holes in F. All closed circuits on F may be reduced to sums of multiples of $2p$ irreducible circuits.

DOUBLE CURVES.

The double curves of the surface F arise as the result of projecting the surface Φ from four space into three space, the center of projection being at infinity. Whenever a projecting line cuts Φ in two places a double point occurs on F. If the two points on Φ be real the double point on F will be a real double point connected with the surface F, but if the two points on Φ be imaginary the resulting double point on F will be isolated. This gives rise to two classes of double curves, one being on the surface F and the other being related to the surface but isolated from it.

In the elliptic case the double curves consisted of the XX axis and an hyperbola. That part of the XX axis included by the real part of the curve $u = f(x)$, $y = 0$ is isolated. Of the hyperbola, that branch lying to the left of the YU plane is isolated.

In the hyper-elliptic example the double curve consists of the XX axis and four infinite branches. What was said of the XX axis for the elliptic case holds here also. Of the four infinite branches two are isolated (see fig. VIII), and two are curves of intersection of the two sheets of the surface.

The same conditions will exist in the general hyper-elliptic case, the XX axis always being a double curve with the same law as to isolated points as in the simpler cases. The other double curves will be partly isolated and partly curves of intersection of the two sheets of the surface. The isolated curves separate themselves from the other class in that they always pass through one or more of the ovals, while the curves of intersection of sheets never do.

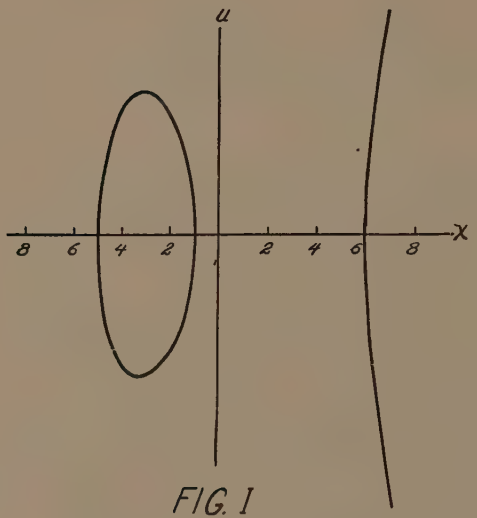

FIG. I

$F(x,y,u)=0,\ y=0$

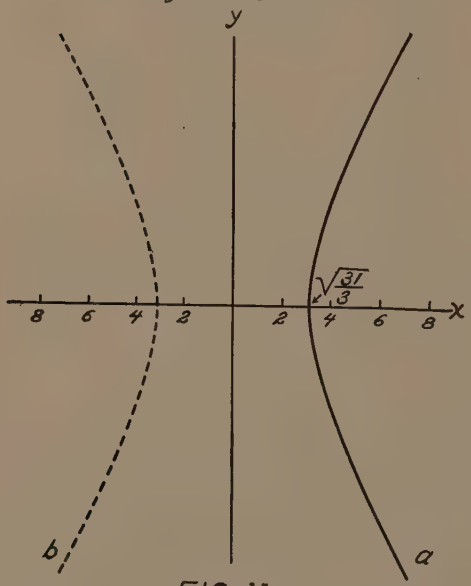

FIG. II

$F(x,y,u)=0,\ u=0$, (b) isolated

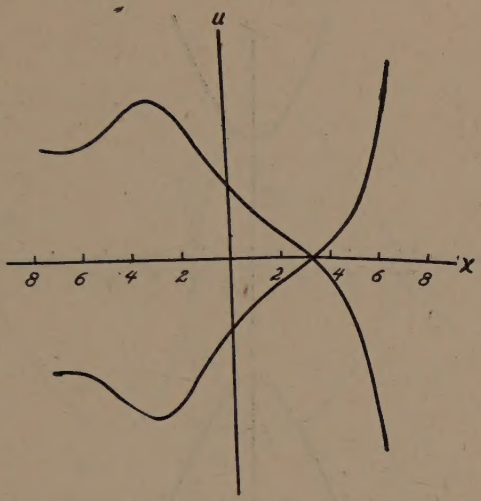

FIG. III

$F(x,y,u)=0, y=1$

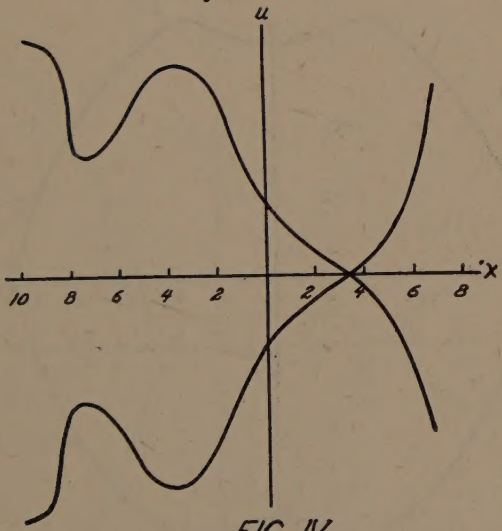

FIG. IV

$F(x,y,u)=0, y=2$

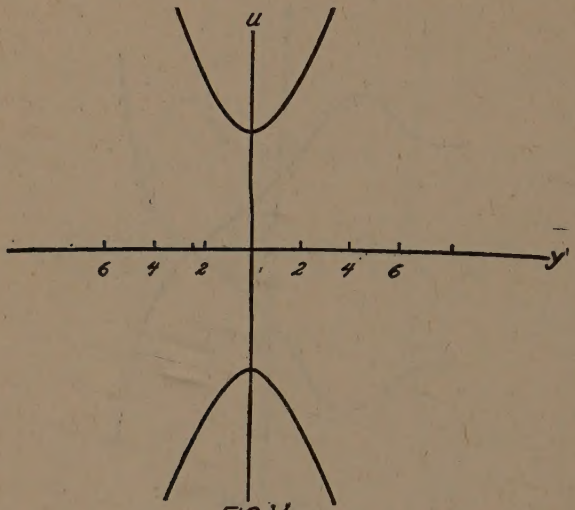

FIG. V
$F(x,y,u)=0,\ x=-2$

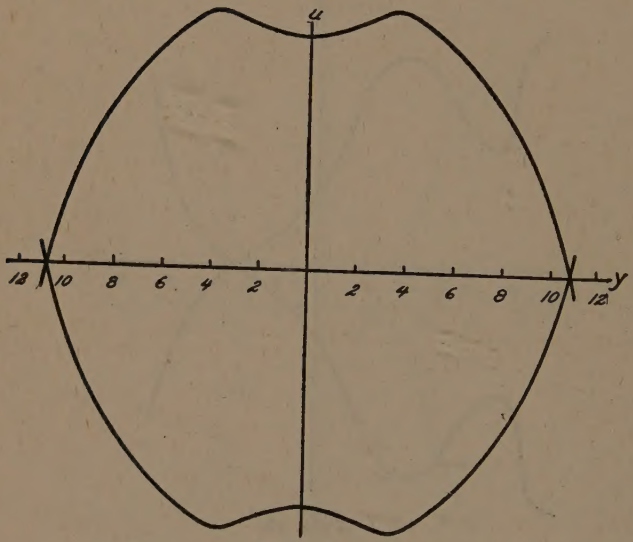

FIG VI
$F(x,y,u)=0,\ x=7$

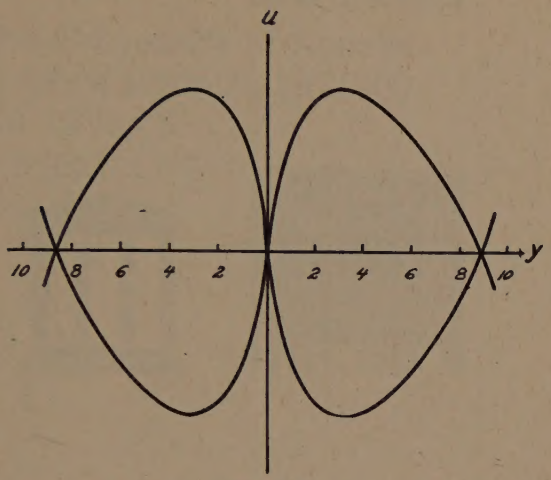

FIG. VII

$F(x,y,u)=0,\ x=6$

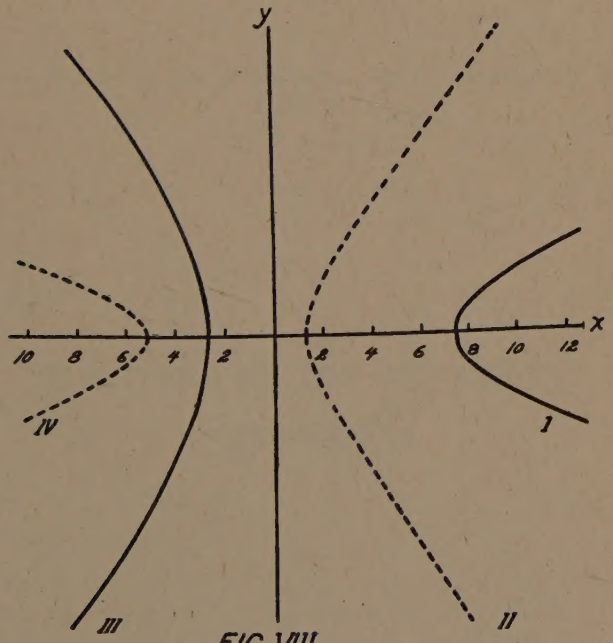

FIG. VIII

$F(x,y,u)=0,\ u=0$ II and IV isolated.